Numerical Results

The cantilever beam with piezoelectric actuator pair shown in Fig. 1 is used to illustrate the procedure just outlined. The piezoelectric material is taken as polyvinylidene fluoride (PVDF), and the beam is made of aluminum. The material properties of PVDF are $c_{11} = 3.8 \times 10^9$ Pa, $e_{31} = 0.046$ C/m^2, $\varepsilon_{33} = 1.062 \times 10^{-10}$ F/m, $c_p = 3.8 \times 10^{-6}$ F, and $\rho = 1800$ kg/m^3; those of aluminum are $c_{11} = 7.3 \times 10^{10}$ Pa, $\rho = 2750$ kg/m^3, and $\nu = 0.33$. The heights of the beam and piezoelectric actuators are taken as $h_b = 6$ and 0.6 mm. The depth of both beam and piezoelectric actuators is $w = 6$ mm, and the length of the beam is $L = 60$ mm. The length of both piezoelectric actuators is assumed as $L/2$.

The closed-loop structure controlled according to the LQR scheme is subjected to a unit step force at the tip in the y direction, and the simulation results are given in Figs. 2 and 3. Figure 2 shows the tip deflection (tip displacement in the y direction) of the beam for two different actuator locations, whereas Fig. 3 shows the absolute sum of the tip deflections of the beam in the given time interval (the sum is found by adding the absolute magnitudes at every 0.001 s). The Q and R matrices for the LQR method are taken as $0.001 \times I$ and I, where I is the identity matrix with proper dimensions. Other Q and R matrices can be used equally well. It is evident from the results, especially in Fig. 3, that the piezoelectric actuator pair closer to the fixed end is more effective in attenuating structural vibrations than the pair further away from the fixed end.

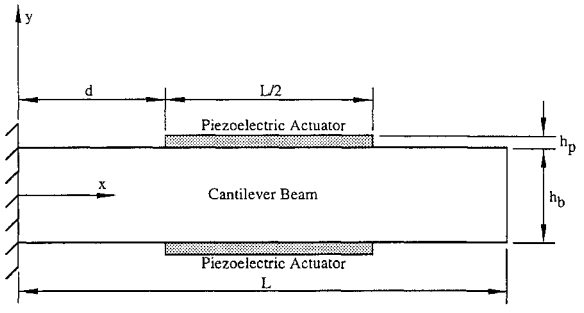

Fig. 1 Cantilever beam with piezoelectric actuator pair (not to scale).

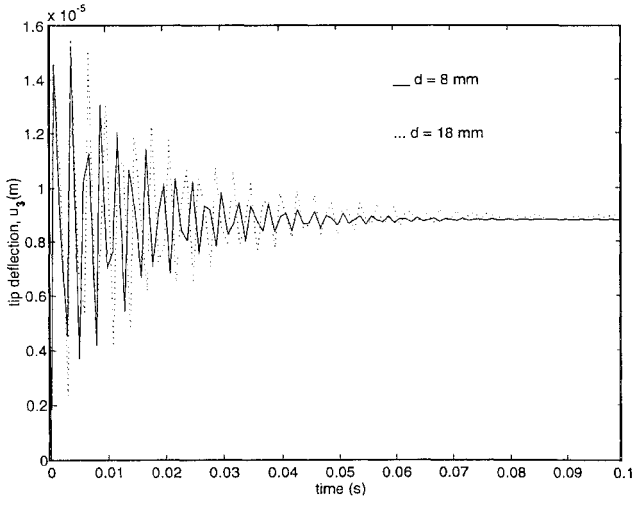

Fig. 2 Vibrational tip deflection of closed-loop system.

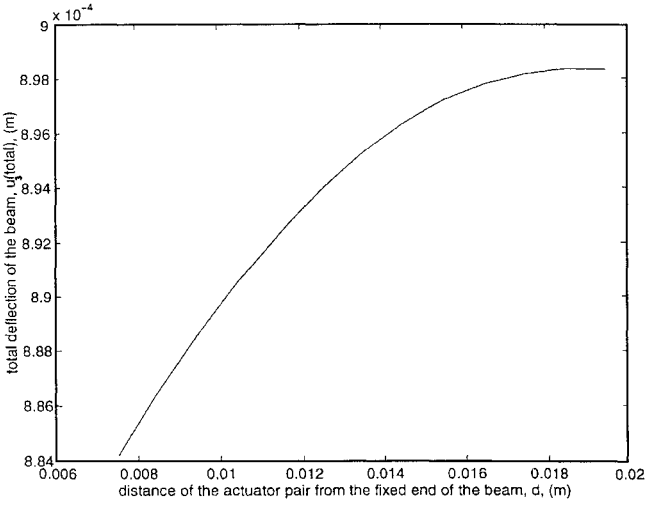

Fig. 3 Absolute sum of vibrational tip deflections of closed-loop system from 0 to 0.1 s.

Conclusion

Observing the magnitude of structural vibrations, the piezoelectric actuator pair closer to the fixed end of the cantilever beam is more efficient than the pair further away from the fixed end. This result is important in piezoelectric actuator placement for structures that can be modeled as cantilever beams. The procedure outlined in this Note can be generalized to more complicated systems.

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Practical Complete Modal Space and Its Applications

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I. Introduction

THE modal expansion method has been used in many important subjects, such as dynamic response, substructural modal synthesis, calculation of eigenvector derivatives, model correction, and reduction of dynamic models. In using the modal expansion technique, there exists numerical error as a result of truncation of modes, and the precision of the results sometimes is poor. To reduce the error of truncated modes, the development of a practical complete modal space (PCMS) method is necessary and strongly recommended.

The PCMS method was first applied to the correction of structural dynamic models.[1–4] Later, it was successfully used in the calculation of the eigenvector derivatives with repeated frequencies,[5] in the development of an accurate modal synthesis,[6–9] in the reduction of dynamic models,[10,11] and in the medium substitution problems for

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test structures.[12] The results of using the PCMS, studied by both authors for many years, are summarized in this Note.

In the PCMS method, the lower-order modes, including rigid-body modes, are obtained from solving the eigenequation. All high-order modes are replaced by equivalent high-order modes that are given by using a simple matrix projection approach. This replacement means that the entire contribution of the equivalent high-order modes is equal to that of the original high-order modes. But the equivalence between the corresponding column vectors of both high-order mode matrices does not exist. The derivation of the equivalence between the entire contribution of both high-order modes is presented in this Note.

Also, there are many techniques published in the literature that can be compared with the PCMS method. For example, with respect to the modal synthesis technique, the free-interface method,[13] the fixed-interface method,[14] and the mixed-interface method[15] are all approximately substructural coupling methods based on the incomplete modal space. Based on the PCMS method, the accurate modal synthesis methods can give any order of modes with very good precision for assembly structure, since the precision of the Rize analysis is determined principally by the completeness of the basis of the vector space.

In the field of the calculation of eigenvector derivatives. Fox and Kapoor[16] developed an incomplete modal expansion technique with poor precision. Wang[17] improved the Fox–Kapoor method by adding the static modes to the modal expansion formula. But their methods merely guarantee that the precision of the eigenvector derivatives of a few lower-order modes is accurate. On the other hand, a complete modal method[5] based on the PCMS method can make the precision of the eigenvector derivatives of many high-order modes accurate.

In regard to the reduction of the dynamic model, Kammer[18] proposed an incomplete modal method that can only guarantee that some lower-order modes of the reduced model are exact inside the frequency range of interest. However, the PCMS method can make available many modes of reduced model outside the range of interest.

There are some limitations of the originally developed PCMS method. The orthogonality relationship between the column vectors of the equivalent matrix of high-order modes and the mass and stiffness matrices does not exist. This limitation reduces the computational efficiency. In addition, this nonorthogonality behavior may sometimes reduce the precision for certain applications of the PCMS method.[10,11] To improve the orthogonality relationship between the equivalent high-order mode matrix and the mass and stiffness matrices, an optimal Gram–Schmidt orthogonalization method can be used in the analysis.

II. Derivation of Practical Complete Modal Space

The eigenequation of a system with n degrees of freedom (DOF) is

$$K\varphi_i = \lambda_i M\varphi_i, \qquad i = 1, 2, \ldots, n \tag{1}$$

in which M is a positive definite mass matrix and K is a positive semidefinite stiffness matrix. $\Lambda_l = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_l]$ and $\Phi_l = [\varphi_1, \varphi_2, \ldots, \varphi_l] \in R^{n,l}$ represent numbers l of lower-order eigenpairs that are computed from Eq. (1) and satisfy $\Phi_l^T M \Phi_l = I$ and $\Phi_l^T K \Phi_l = \Lambda_l$. Thus, it is known that

$$M^{-1} = \Phi_l \Phi_l^T + \Phi_h \Phi_h^T \tag{2}$$

in which $\Phi_h = [\varphi_{l+1}, \ldots, \varphi_n] \in R^{n,h} (h = n - l)$ are mass-normalized high-order modes of Eq. (1); i.e., that $\Phi_h^T M \Phi_h = I$ and $\Phi_h^T K \Phi_h = \Lambda_h$. Note that Φ_h and $\Lambda_h = \text{diag}[\lambda_{l+1}, \ldots, \lambda_n]$ do not need to be calculated from Eq. (1). From Eq. (2) we have

$$\Phi_h \Phi_h^T M = I - \Phi_l \Phi_l^T M = \bar{\Psi}_h (\in R^{n,n}) \tag{3}$$

The rank of matrix $(\Phi_h \Phi_h^T)$ is h and the rank of M is n; therefore, the rank of matrix $\bar{\Psi}_h$ is equal to h, where $h = n - l$. This denotes that there are only h independent columns in the matrix $\bar{\Psi}_h$, although these h independent columns are nonunique in the matrix $\bar{\Psi}_h$. There are always l dependent columns in the matrix $\bar{\Psi}_h$. From the mathematical point of view, one can adopt any numerical analysis technique to eliminate l dependent columns from the matrix $\bar{\Psi}_h$ to obtain an equivalent high-order modal matrix $\Psi_h \in R^{n,h}$. The term Φ_h can be substituted by Ψ_h to form PCMS $[\Phi_l, \Psi_h]$.

It is demonstrated that

$$\Phi_l^T M \Psi_h = 0, \qquad \Phi_l^T K \Psi_h = 0 \tag{4}$$

For this,

$$\Phi_l^T M \bar{\Psi}_h = 0, \qquad \Phi_l^T K \bar{\Psi}_h = 0 \tag{5}$$

are first proven. Embedding Eq. (3) into Eq. (5) gives

$$\begin{aligned} \Phi_l^T M \left(I - \Phi_l \Phi_l^T M \right) &= \Phi_l^T M - \Phi_l^T M = 0 \\ \Phi_l^T K \left(I - \Phi_l \Phi_l^T M \right) &= \Phi_l^T K - \Lambda_l \Phi_l^T M = 0 \end{aligned} \tag{6}$$

Because h columns of Ψ_h are included inside n columns of $\bar{\Psi}_h$, Eq. (4) does exist. Similarly, we can demonstrate

$$\begin{aligned} \mu_h &= \Psi_h^T M \Psi_h \neq I \quad \text{or} \quad \text{diagonal matrix} \\ \kappa_h &= \Psi_h^T K \Psi_h \neq \Lambda_h \quad \text{or} \quad \text{diagonal matrix} \end{aligned} \tag{7}$$

Other properties of $\bar{\Psi}_h$ are specifically provided here that are easy for readers to follow:

$$\bar{\Psi}_h \bar{\Psi}_h \cdots \bar{\Psi}_h = \bar{\Psi}_h \tag{8}$$

$$\bar{\Psi}_h \Phi_l = 0 \tag{9}$$

Substituting Eq. (3) into Eqs. (8) and (9) shows that Eqs. (8) and (9) do exist; for example, substituting Eq. (3) into Eq. (9) gives $\Psi_h \Phi_l = (I - \Phi_l \Phi_l^T M)\Phi_l = \Phi_l - \Phi_l = 0$. The equivalence between PCMS $[\Phi_l, \Psi_h]$ and the original complete modal space $[\Phi_l, \Phi_h]$ can be mathematically proven. This equivalence is described through the equivalence of two mathematical models (M, K) in two different kinds of the aforementioned complete space. The proven demonstration is made in terms of the orthogonalization conditions, because the mathematical model that satisfies the orthogonal relation must satisfy the eigenequation using the complete space. For the sake of simplicity, the demonstration is progressed merely in terms of the M-orthogonalization condition, since the process of the demonstration made by the K-orthogonalization relation is the same. The M-orthogonality condition in the original space is

$$[\Phi_l, \Phi_h]^T M [\Phi_l, \Phi_h] = I \tag{10}$$

Equation (10) can be rewritten as

$$\Phi_l^T M \Phi_l = I \tag{11a}$$

$$\Phi_h^T M \Phi_l = 0 \tag{11b}$$

$$\Phi_h^T M \Phi_h = I \tag{11c}$$

Clearly, the following expressions based on Eqs. (11b) and (11c) can be obtained:

$$\Phi_h \Phi_h^T M \Phi_l = 0 \tag{12a}$$

$$\Phi_h \Phi_h^T M \Phi_h \Phi_h^T = \Phi_h \Phi_h^T \tag{12b}$$

Because M is a positive matrix, Eq. (12) must be equivalent to the following formulas:

$$M \Phi_h \Phi_h^T M \Phi_l = 0 \tag{13a}$$

$$M \Phi_h \Phi_h^T M \Phi_h \Phi_h^T M = M \Phi_h \Phi_h^T M \Phi_h \Phi_h^T M \tag{13b}$$

From Eqs. (3) and (7) that define Ψ_h and μ_h, Eq. (13) can be rewritten as

$$\Psi_h^T M \Phi_l = 0 \tag{14a}$$

$$\Psi_h^T M \Psi_h = \mu_h \tag{14b}$$

Table 1 Error analysis of the synthetic eigenpairs

Mode	Eigenvalue of substructure $\lambda^{\alpha,\beta}$	Eigenvalue of assembly structure $\bar{\lambda}$	Synthetic eigenpairs of assembly structure λ	λ%	$\|x\|$%
1	0.37214D−7	0.82825D−8	0.53593D−5 (0.43250D−10)	——	——
2	0.14179D−6	0.21648D−7	0.81490D−5 (0.45978D−6)	——	——
3	0.79996D+5	0.49998D+4	0.49996D+4	0.450D−4	0.507D−4
4	0.60809D+6	0.37990D+5	0.37990D+5	0.157D−2	0.654D−3
5	——	0.14602D+6	0.14601D+6	0.396D−4	0.432D−4
6	——	0.39908D+6	0.39908D+6	0.658D−6	0.602D−6
7	——	0.89090D+6	0.89090D+6	0.487D−4	0.154D−3
8	——	0.17391D+7	0.17392D+7	0.318D−2	0.164D−1
9	——	0.30859D+7	0.30859D+7	0.547D−3	0.622D−1
10	——	0.50986D+7	0.50989D+7	0.466D−2	0.176D−1
11	——	0.79730D+7	0.79731D+7	0.129D−2	0.203D−0
12	——	0.11932D+8	0.11926D+8	0.529D−1	0.759D−0

Thus Eq. (15) is obtained from Eqs. (11a) and (14):

$$[\Phi_l, \Psi_h]^T M[\Phi_l, \Psi_h] = \begin{bmatrix} I & 0 \\ 0 & \mu_h \end{bmatrix} \tag{15}$$

The aforementioned derivation shows that if Eqs. (12a) and (12b) are equivalent to Eqs. (11b) and (11c), respectively, then Eq. (15) must be equivalent to Eq. (10). The procedure of the demonstration is listed as follows:

1) To prove the equivalence between Eqs. (12a) and (11b): Let Eq. (12a) have a solution M that does not satisfy Eq. (11b). There is at least a nonzero column vector q in $\Phi_h^T M \Phi_l$, i.e., $q \neq 0$. However, from Eq. (12a) we get $\Phi_h q = 0$. Because all column vectors of Φ_h are linearly independent, q has to be a zero vector. This is contradictory to the original assumption.

2) To prove the equivalence between Eqs. (12b) and (11c):

a) First, to prove that Eq. (11c) is equivalent to

$$\Phi_h \Phi_h^T M \Phi_h = \Phi_h \tag{16}$$

let Eq. (16) have a solution M that does not satisfy Eq. (11c). Then, there is at least a column vector p_i in $\Phi_h^T M \Phi_h$ that is not a unit vector e_i, i.e., $p_i \neq e_i$. Thus we obtain from Eq. (16)

$$\Phi_h p_i = \varphi_{h,i} \tag{17}$$

where $\varphi_{h,i}$ denotes the ith column of Φ_h. Obviously, when $p_i \neq e_i$, Eq. (17) is contradictory to the fact that all column vectors of Φ_h are linearly independent among each other, since Eq. (17) denotes that an arbitrary column of Φ_h is the linear combination of all columns of Φ_h. Thus, there must exist $p_i = e_i$.

b) Second, there exists the relation from Eq. (16)

$$\Phi_h^T M \Phi_h \Phi_h^T = \Phi_h^T \tag{18}$$

To demonstrate the equivalence between Eq. (18) [i.e., Eq. (16)] and Eq. (12b), suppose that Eq. (12b) has a solution M that does not suit Eq. (18), and that there is at least a column p in $\Phi_h^T M \Phi_h \Phi_h^T$, which is not the same as the corresponding column q of Φ_h^T, i.e., $p \neq q$, and then we have

$$\Phi_h p \neq \Phi_h q \tag{19}$$

Clearly, Eq. (19) is contradictory to Eq. (12b).

III. Applications: Accurate Free-Interface Modal Synthesis

Employing PCMS $[\Phi_l, \Psi_h]$ as Rize space can set up accurate modal syntheses of substructures with various interfaces, for example, free interface,[6,7] fixed interface,[8] and mixed interface,[9] etc. Modal synthesis techniques are based on Rize analysis. Rize analysis tells us that the accuracy of the results depends primarily on the completeness of the base vector space. Hence, even if the shape of Ψ_h does not have the approximation of the shape of Φ_h, the total contribution of Ψ_h can be equivalent to that of Φ_h. For this reason, the accurate free-interface method described here is based on the PCMS and can acquire the synthetic results with high precision (see Table 1).

The numerical example of Table 1 is a free bending uniform beam. Its characteristics are total length $L = 100$, area $A = 1.0$, moment of inertia $I_Z = 1.0$, Young's modulus $E = 1.0 \times 10^6$, and mass density $\rho = 1.0 \times 10^{-3}$. The finite element method of this beam is combined by 20 identical beam finite elements. Every node has 2 DOFs, and the DOFs of the whole beam are 42. Two identical substructures (α and β) are formed by dividing the free beam at the middle point so that every substructure has 22 DOFs. Only the first lower modes $\Phi_l \in R^{22,4}$ (two rigid-body modes and two elastic modes) of every substructure are calculated by solving the eigenequation. The contribution of the other 18 high-order modes is equivalent to that of $\Psi_h \in R^{22,18}$. The dimension of the synthetic eigenequation is of order 8×8. For explaining that all $n = 42$ eigenpairs of the entire beam can be synthesized from the eigenequation of order 8×8, the first 12 characteristic pairs of entire beam are computed by using this synthetic eigenequation.

IV. Concluding Remarks

After the PCMS technique was investigated systematically and successfully applied to many fields, it was concluded that any techniques that need to utilize the modal expansion procedure will make possible use of the PCMS so that the accuracy of the results can be increased. Of course, the Ψ_h presented in this Note is only one type of solution method to find the PCMS. The best solution method is still under study. The authors strongly recommend that in further development of the method the best Ψ_h should be found to make both μ_h and κ_h be diagonal matrices.

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Computation of Derivatives of Repeated Eigenvalues and Corresponding Eigenvectors by Simultaneous Iteration

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I. Introduction

THE past 30 years have seen much effort[1] devoted to the development of numerical methods for computing partial derivatives (sensitivities) of eigenvalues and eigenvectors of matrices, which depend smoothly on a number of real-valued design parameters, $\rho_1, \ldots, \rho_m$. These derivatives are important in the optimum design of structures,[2] in mode tracking[3] and in model updating.[4] Until recently,[1,5–7] most of this work was restricted to the case of simple (i.e., nonrepeated) eigenvalues, although it is well known that eigenvalues often coalesce as a design structure approaches an optimum,[2,8] and, even before optimizing, repeated eigenvalues may occur when a structure has certain symmetry properties.[9] Also, as previously noted,[10] some earlier work on the repeated-eigenvalue case is flawed.

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This Note presents a new algorithm for computing partial derivatives of repeated eigenvalues and the corresponding eigenvectors. Formally, we consider the problem

$$A(\rho)x_i(\rho) = \lambda_i(\rho)x_i(\rho), \qquad i = 1, \ldots, n \tag{1}$$

where $A(\rho)$ is a nondefective $n \times n$ matrix depending smoothly on $\rho = (\rho_1, \ldots, \rho_m) \in \mathbb{R}^m$ and $\lambda_i(\rho)$ and $x_i(\rho)$ are eigenvalues and the corresponding eigenvectors. We denote first- and second-order partial derivatives with respect to ρ_j by the subscripts j and jj, respectively. Our algorithm computes $\lambda_{i,j}(\rho_0)$, $\lambda_{i,jj}(\rho_0)$, and $x_{i,j}(\rho_0)$, $i = 1, \ldots, r$, when $\lambda_1(\rho_0) = \cdots = \lambda_r(\rho_0)$ and, for some constant σ, $|\lambda_1(\rho_0) - \sigma| > |\lambda_{r+1}(\rho_0) - \sigma| \geq \cdots \geq |\lambda_n(\rho_0) - \sigma|$, i.e., the r dominant eigenvalues (or r eigenvalues that can be made dominant by a suitable origin shift[11–13]) coincide at the point $\rho = \rho_0$.

Computation of derivatives of eigenvalues and (especially) eigenvectors is relatively difficult for repeated eigenvalues, partly because of the nonuniqueness of the corresponding eigenvectors.[7] (The even more difficult problem[14] for defective matrices is not considered here.) Differentiability of the eigenvectors requires that an appropriate basis be chosen for the multidimensional eigenspace corresponding to the repeated eigenvalue. This basis will not generally be known a priori. Moreover, there are well-known examples[2,8,9,15] in which, regardless of the choice of this basis, the repeated eigenvalues and the corresponding eigenvectors are not differentiable. Nevertheless, there is also a large class of important examples[15] in which repeated eigenvalues and the corresponding eigenvectors are not only differentiable but also analytic. This Note considers only the case in which all required derivatives exist.

Our main result is the previously promised[13,16] extension of our simultaneous iteration method[12] (sometimes called subspace iteration) for computing partial derivatives of eigenvalues and eigenvectors to the case where the dominant eigenvalue is repeated. We remarked previously[12,13] that the methods described in those papers could be used to compute derivatives of repeated eigenvalues (though not the corresponding eigenvectors). However, this remark was made in the context of the assumption that the eigenvectors were differentiable. As noted above, this requires a particular choice of basis for the eigenspace at $\rho = \rho_0$. This choice was indeed used in the numerical calculations reported previously.[13]

We use an idea of Mills-Curran,[7] who considered only the symmetric case, to modify the simultaneous iteration method[12] so that an arbitrary basis for the eigenspace corresponding to $\lambda_1(\rho_0) = \cdots = \lambda_r(\rho_0)$ can be used to compute $\lambda_{i,j}(\rho_0)$, $\lambda_{i,jj}(\rho_0)$, and $x_{i,j}(\rho_0)$, $i = 1, \ldots, r$, provided the $\lambda_{i,j}(\rho_0)$ are well separated. Using techniques that have been developed for direct methods,[5] our algorithm can be modified to deal with problems where the eigenvalue derivatives also are repeated. Although an iterative method is available for the simpler problem of computing derivatives of multidimensional invariant subspaces,[17] previous iterative methods for computing derivatives of individual eigenvectors[11–13,16,18,19] require that the corresponding eigenvalues be simple. When comparing our iterative method with direct[5,7] or modal-expansion[6] methods, note that the extrapolation methods previously used for simple eigenvalues[12,13,16,18] also can be used with the algorithm presented here for repeated eigenvalues, and this can increase its efficiency dramatically.

Our new algorithm is described and a summary (Theorem 1) of its main properties is given in Sec. II; Sec. III illustrates the algorithm by a simple example. More theoretical matters will be addressed in a later paper.[20] These include 1) a proof of Theorem 1 and an analysis of the rate of convergence of the iteration; 2) extension of Algorithm 1 to the case in which the $\lambda_{i,j}(\rho_0)$ are not well separated and to problems with repeated subdominant eigenvalues and to the computation of higher-order derivatives[13]; 3) consideration of questions of existence of derivatives; 4) theory of extrapolation procedures required for a valid comparison with other methods; and 5) an analysis of the numerical stability of the algorithm.

II. Simultaneous-Iteration Algorithm

The derivatives of eigenvectors depend on the normalizing condition used.[15] For definiteness, we consider the case

$$x_i^*(\rho)x_i(\rho) = 1, \qquad i = 1, \ldots, n \tag{2}$$